Generator: $(P_h)_{ij} \triangleq \mathbb{P}(Y_h = j \mid Y_0 = i)$ for time-homo CTMC $\{Y_t\}$, is called the semi-group operator

check: $\begin{cases} P_{h_1} P_{h_2} = P_{h_1+h_2} = P_{h_2} P_{h_1} \\ P_0 = I \end{cases}$

generator $G \triangleq \lim_{h \to 0} \frac{P_h - I}{h}$ $\quad \left.\frac{d}{dh} P_h\right|_{h=0}$

Calculation: $G_{ij} = \begin{cases} -q_i & \text{if } i = j \\ q_i P_{ij} & \text{else} \end{cases}$

provides "rate" interpretation of CTMC!

$$\begin{cases} P_{ii}(h) = 1 + \underbrace{G_{ii}}_{\text{neg flow rate out of } i} h + o(h) \\ \forall i \neq j, \; P_{ij}(h) = \underbrace{G_{ij}}_{\text{flow rate } i \to j} h + o(h) \end{cases} \quad (h \to 0)$$

so $G_{ii} + \sum_{j \neq i} G_{ij} = 0$ for $\forall i$

(row sums up to 0)

Recover P_t from G?

$$\begin{cases} P_t' = G P_t = P_t G & \text{(forward/backward eqn)} \\ P_0 = I \end{cases}$$

G and P has 1-to-1 correspondence, uniquely characterizes CTMC.

exponential analogue:
$$\begin{cases} y' = ay \Rightarrow y = c \cdot e^{at} \\ P_t' = G P_t \Rightarrow P_t = e^{tG} \end{cases}$$

What happens if MC has cts state?
Then matrix dimension becomes uncountable, instead of a seq of matrix P_t, we have a seq of function $p(t, x)$, forward / backward eqn becomes PDE!

x: space variable

eg: $\lambda\mu>0$, MC on $\{1,2\}$, $G=\begin{pmatrix}-\mu & \mu \\ \lambda & -\lambda\end{pmatrix}$

(a): Forward eqn, solve for $P_{ij}(t)$

Pf: $P_t=\begin{pmatrix}P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t)\end{pmatrix}$, $P_t'=GP_t$ writes

$$\begin{cases} P_{11}'(t) = -\mu P_{11}(t)+\lambda P_{12}(t) \\ P_{12}'(t) = \mu P_{11}(t)-\lambda P_{12}(t) \\ P_{21}'(t) = -\mu P_{21}(t)+\lambda P_{22}(t) \\ P_{22}'(t) = \mu P_{21}(t)-\lambda P_{22}(t) \end{cases} \quad \text{with } P_{ij}(0)=\delta_{ij}$$

Since $P_{11}(t)+P_{12}(t)=1$, $\begin{cases} P_{11}'(t)=-(\lambda+\mu)P_{11}(t)+\lambda \\ P_{11}(0)=1 \end{cases}$

$$\Downarrow$$

$$P_{11}(t)=\frac{\lambda}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

$$\Downarrow$$

$$P_{12}(t)=\frac{\mu}{\lambda+\mu}-\frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

Similarly, $P_{21}(t)=\frac{\lambda}{\lambda+\mu}-\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$

$$P_{22}(t)=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

(b): Calculate G^n and $\underbrace{\sum_{n=0}^{\infty} \frac{t^n}{n!} G^n}_{\text{Def of } e^{tG} \text{ for operator } G}$, compare to part (a).

Pf:

G has eigval 0 and $-(\lambda+\mu)$, with eigvec

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} \mu \\ -\lambda \end{pmatrix}$$

So $G = PDP^{-1}$, $P = \begin{pmatrix} 1 & \mu \\ 1 & -\lambda \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & -(\lambda+\mu) \end{pmatrix}$

$$P^{-1} = \begin{pmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{1}{\lambda+\mu} & -\frac{1}{\lambda+\mu} \end{pmatrix}$$

$$G^n = PD^nP^{-1} = [-(\lambda+\mu)]^n \cdot \begin{pmatrix} \frac{\mu}{\lambda+\mu} & -\frac{\mu}{\lambda+\mu} \\ -\frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix} \quad (n \geq 1)$$

$$e^{tG} = \sum_{n=1}^{\infty} \frac{[-(\lambda+\mu)t]^n}{n!} \begin{pmatrix} \frac{\mu}{\lambda+\mu} & -\frac{\mu}{\lambda+\mu} \\ -\frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix} + I$$

$$= \begin{pmatrix} \frac{\mu}{\lambda+\mu} & -\frac{\mu}{\lambda+\mu} \\ -\frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix} \cdot e^{-(\lambda+\mu)t} + I - \begin{pmatrix} \frac{\mu}{\lambda+\mu} & -\frac{\mu}{\lambda+\mu} \\ -\frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\mu}{\lambda+\mu} & -\frac{\mu}{\lambda+\mu} \\ -\frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix} \cdot e^{-(\lambda+\mu)t} + \begin{pmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{pmatrix}$$

$$= P_t \checkmark$$

(c): Find stat dist π. Check $P_{ij}(t) \to \pi_j$ $(t \to \infty)$

Pf: By def, $\forall t$, $\pi P_t = \pi$, $\pi(P_t - I) = 0$,

so divide by t and set $t \to 0$ to get

$$\pi G = 0$$

$$\Downarrow$$

$$\pi = \left(\frac{\lambda}{\lambda+\mu} \quad \frac{\mu}{\lambda+\mu} \right)$$

and $P_{ij}(t) \to \pi_j$ $(t \to \infty)$ ✓

Consistent with the ergodic thm.

(d): Calculate $\mathbb{P}(X_t = 2 \mid X_0 = 1, X_{3t} = 1)$

Pf:

$$= \frac{\mathbb{P}(X_{3t}=1 \mid X_0=1, X_t=2) \cdot \mathbb{P}(X_t=2 \mid X_0=1)}{\mathbb{P}(X_{3t}=1 \mid X_0=1)}$$

$$\overset{\text{Markov}}{=} \frac{P_{2,1}(2t) \cdot P_{1,2}(t)}{P_{1,1}(3t)}$$

$$\overset{\text{use (a)}}{=} \frac{\left[\frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)2t}\right] \left[\frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}\right]}{\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)3t}}$$

Applications of CTMC:

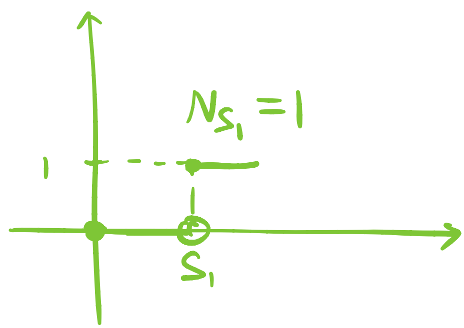
e.g.: $\{N_t\}$ is Poisson process (intensity λ), $N_0 = 0$, $\phi(t,z) \triangleq \mathbb{E} z^{N_t}$ is g.f. of N_t, find an integral equation for $\phi(t,z)$ and check that $\phi(t,z) = e^{\lambda t(z-1)}$ is the solution.

Pf: Poisson process is CTMC, pure birth process with birth rate $\lambda_i = \lambda$ at any state i.

By first step decomposition, let S_1 be the first time a state transition happens.

$$\phi(t,z) = \mathbb{E}\left[\mathbb{E}(z^{N_t} \mid S_1)\right]$$

with $\mathbb{E}(z^{N_t} \mid S_1 = s)$



$$= \mathbb{E}(z^{N_s} \cdot z^{N_t - N_s} \mid S_1 = s)$$

$$= z \cdot \underbrace{\mathbb{E}(z^{N_t - N_s} \mid S_1 = s)}$$

Markov property, $t-s$ time left, a single particle at time S_1

$$= z \cdot \mathbb{E} z^{N_{t-s}} = z \cdot \phi(t-s, z) \quad \text{if } s < t$$

So: $\mathbb{E}(z^{N_t} | S_1) = z \cdot \phi(t - S_1, z)$ if $S_1 < t$

When taking expectation w.r.t. S_1, notice that S_1 may not be well-defined since $S_1 \sim \mathcal{E}(\lambda)$ and it's possible that no transition has happened up to time t.

So $\underline{\phi(t,z)} = \mathbb{E}\Big(z^{N_t} \cdot \underbrace{I_{\{S_1 > t\}}}_{\text{under this event, } N_t = 0}\Big) + \mathbb{E}\left(z^{N_t} \cdot I_{\{S_1 < t\}}\right)$

$$= \mathbb{P}(S_1 > t) + z \cdot \mathbb{E}\left[\phi(t - S_1, z) \cdot I_{\{S_1 < t\}}\right]$$

$$= \underline{e^{-\lambda t} + z \cdot \int_0^t \phi(t-s, z) \cdot \lambda e^{-\lambda s}\, ds}$$

provides an integral eqn for ϕ.

<u>Check</u>: $\phi(t,z) = e^{\lambda t(z-1)}$ is solution.

$$\text{RHS} = e^{-\lambda t} + z \cdot e^{\lambda(z-1)t} \cdot \int_0^t \lambda \cdot e^{-\lambda s z}\, ds$$

$$= e^{-\lambda t} + z \cdot e^{\lambda(z-1)t} \cdot \frac{1}{z}\left(1 - e^{-\lambda t z}\right)$$

$$= e^{\lambda(z-1)t} = \phi(t,z) \checkmark$$

RMK: Why last step decomposition does not work?

Let T be last transition time before t, condition on T brings difficulty since N_T is unknown except $N_T + 1 = N_t$ while in first time decomp~ $N_{S_1} = N_0 + 1 = 1$.

both random

e.g.: $\{X_t\}$ is cts-time BDC on $\mathbb{N}$ with birth rate λ_n death rate μ_n, assume $X_0 = i$, let T_{i+1} be first hitting time to $i+1$. Derive a recursive formula calculating

$$m_i \triangleq \mathbb{E}_i T_{i+1}.$$

Pf: Still use first step decomposition, let S_1 be the first transition time, $S_1 \sim \mathcal{E}(\lambda_i + \mu_i)$

since $X_0 = i$

$$m_i = \mathbb{E}_i T_{i+1} = \mathbb{E}_i\left[\mathbb{E}_i(T_{i+1} \mid S_1)\right]$$

where $\mathbb{E}_i(T_{i+1} \mid S_1 = s)$ X_{S_1} could take value $i-1$ or $i+1$ w.p. $\frac{\mu_i}{\lambda_i+\mu_i}$ or $\frac{\lambda_i}{\lambda_i+\mu_i}$

$$= \mathbb{P}_i(N_{S_1} = i+1 \mid S_1 = s) \cdot \mathbb{E}_i(T_{i+1} \mid S_1 = s, N_{S_1} = i+1) +$$

$$\mathbb{P}_i(N_{S_1} = i-1 \mid S_1 = s) \cdot \mathbb{E}_i(T_{i+1} \mid S_1 = s, N_{S_1} = i-1)$$

Markov property

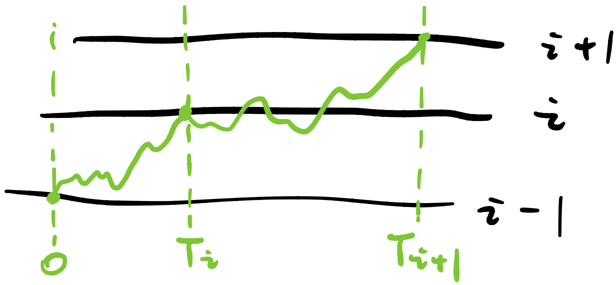
$$= \frac{\lambda_i}{\lambda_i+\mu_i} \cdot s + \frac{\mu_i}{\lambda_i+\mu_i} \cdot (s + \mathbb{E}_{i-1} T_{i+1})$$

$$= s + \frac{\mu_i}{\lambda_i+\mu_i} \underline{\mathbb{E}_{i-1} T_{i+1}} \Longrightarrow$$

can't represent using $\{m_k\}$

need further simplification!

Structure of BDC implies that if hits $i+1$ starting from $i-1$, one must first hit i.



$$\mathbb{E}_{i-1} T_{i+1} = \mathbb{E}_{i-1} T_i + \mathbb{E}_{i-1}(T_{i+1} - T_i)$$

$$= \mathbb{E}_{i-1} T_i + \mathbb{E}_{i-1}\left[\mathbb{E}_{i-1}(T_{i+1} - T_i \mid T_i)\right]$$

$$= \mathbb{E}_{i-1} T_i + \mathbb{E}_{i-1}\left[\underbrace{\mathbb{E}_{i-1}(T_{i+1} \mid T_i)}_{\text{Strong Markov property}} - T_i\right]$$

$$= \mathbb{E}_{i-1} T_i + \mathbb{E}_i T_{i+1}$$

Strong Markov property

$$= \underbrace{T_i}_{\text{already waited}} + \underbrace{\mathbb{E}_i T_{i+1}}_{\text{to wait in the future}}$$

$$S_0: \mathbb{E}_i(T_{i+1} \mid S_1 = s) = s + \frac{\mu_i}{\lambda_i + \mu_i}(m_{i-1} + m_i)$$

$$m_i = \mathbb{E} S_1 + \frac{\mu_i}{\lambda_i + \mu_i}(m_{i-1} + m_i)$$

$$\Downarrow$$

$$\begin{cases} m_i = \dfrac{1}{\lambda_i} + \dfrac{\mu_i}{\lambda_i} m_{i-1} \\ m_0 = \dfrac{1}{\lambda_0} \end{cases}$$

eg: $\begin{cases} \text{sunny} \to \text{wildfire occurs with rate } 0.5 \text{ per day} \\ \text{cloudy} \to \text{———— - ———— } 0.1 \text{ ———— (Poisson)} \end{cases}$

weather: two-state CTMC, sunny lasting on average 2 days, cloudy lasts on average 1 day. Wildfire occurs at time $T_1, T_2, \dots$, as arrival times of a cts-time counting process and weather indep of wildfire occuring.

$F_t \stackrel{\triangle}{=}$ # of fires up to time t.

(i): Compute $\lim\limits_{t\to\infty} \frac{\mathbb{E}F_t}{t}$

Pf: Modelling: S_t is weather at time t, state space $\{s, c\}$, only need to specify holding rates q_s and q_c, clearly,

$$\frac{1}{q_s} = 2, \quad \frac{1}{q_c} = 1, \quad \text{so } q_s = \frac{1}{2}, \ q_c = 1.$$

$\{S_t\}$ irreducible, by ergodic thm, $S_t \xrightarrow{d} \pi$ $(t \to \infty)$ with π as stat dist.

Write out generator matrix
$$G = \begin{array}{c|cc} & s & c \\ \hline s & -\frac{1}{2} & \frac{1}{2} \\ c & 1 & -1 \end{array}$$

so $\pi G = 0 \implies \pi = \left(\frac{2}{3}, \frac{1}{3}\right)$

When $t \to \infty$, $\frac{2}{3}$ days are sunny, $\frac{1}{3}$ days are cloudy.

So: as $t \to \infty$, $\mathbb{E}F_t \sim \frac{2}{3} \cdot \underbrace{(0.5t)} + \frac{1}{3} \cdot \underbrace{(0.1t)}$

PP with $\lambda = 0.5$
$F_t \sim \mathcal{P}(0.5t)$
$\mathbb{E}F_t = 0.5t$

PP with $\lambda = 0.1$

$$\lim_{t \to \infty} \frac{\mathbb{E}F_t}{t} = \frac{2}{3} \cdot 0.5 + \frac{1}{3} \cdot 0.1 = \frac{11}{30}.$$

(2): Explain why $X_t = (S_t, F_t)$ is Markov and write down its generator matrix.

Pf: Given $X_t = (s, n)$, next transition can only be $\to (c, n)$ or $\to (s, n+1)$, depends on which happens first.

Weather s has holding time $\sim \mathcal{E}(q_s)$, fire number n has holding time $\sim \mathcal{E}(0.5)$ when weather is sunny, so holding time of (s, n) is minimum of both,

$$\sim \mathcal{E}(q_s + 0.5)$$

it's still exponentially dist, and different holding times are independent, so it's still CTMC.

Intuitively, $\{F_t\}$ is Poisson process with S_t-dependent intensity, forming as tuple provides Markov prop.

In addition, $\begin{cases} q_{(s,n)} = q_s + 0.5 = 1 \\ q_{(c,n)} = q_c + 0.1 = 1.1 \end{cases}$ are holding rates of $\{X_t\}$.

Still need to figure out the transition prob of the underlying discrete-time MC:

$$\begin{cases} P_{(s,n),(c,n)} = \dfrac{q_s}{q_{(s,n)}} = \dfrac{1}{2} \\ P_{(s,n),(s,n+1)} = 1 - \dfrac{1}{2} = \dfrac{1}{2} \\ P_{(c,n),(s,n)} = \dfrac{q_c}{q_{(c,n)}} = \dfrac{10}{11} \\ P_{(c,n),(c,n+1)} = 1 - \dfrac{10}{11} = \dfrac{1}{11} \end{cases}$$

By def of generator,

$$G^X = \begin{array}{c} \\ (s,0) \\ (s,1) \\ \vdots \\ (c,0) \\ (c,1) \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} (s,0) & (s,1) & (s,2) & \cdots & (c,0) & (c,1) & (c,2) & \cdots \end{array} \\ \begin{pmatrix} -1 & \frac{1}{2} & 0 & \cdots & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & -1 & \frac{1}{2} & \cdots & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & \cdots & -1.1 & 0.1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0 & -1.1 & 0.1 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{pmatrix} \end{array}$$

(3): If it's sunny now, find expected time until next fire $\mathbb{E}_{(s,0)} T_1$.

Pf: First step decomposition, condition on first state transition time S_1 of $\{X_t\}$.

$$\mathbb{E}_{(s,0)} T_1 = \mathbb{E}_{(s,0)}\left[\mathbb{E}_{(s,0)}(T_1 \mid S_1)\right]$$

where $$\mathbb{E}_{(s,0)}(T_1 \mid S_1 = s) = \underbrace{\tfrac{1}{2}}_{P_{(s,0),(s,1)}} \cdot s + \underbrace{\tfrac{1}{2}}_{P_{(s,0),(c,0)}} \cdot \left(s + \mathbb{E}_{(c,0)} T_1\right)$$

$$= s + \tfrac{1}{2}\mathbb{E}_{(c,0)} T_1$$

So: $\mathbb{E}_{(s,0)} T_1 = 1 + \frac{1}{2}\mathbb{E}_{(c,0)} T_1$ $\qquad S_1 \sim \mathcal{E}(q_{(s,0)}) = \mathcal{E}(1)$ if $X_0 = (s,0)$

On the other hand,

$$\mathbb{E}_{(c,0)} T_1 = \mathbb{E}_{(c,0)}\left[\mathbb{E}_{(c,0)}(T_1 \mid S_1)\right]$$

where $$\mathbb{E}_{(c,0)}(T_1 \mid S_1 = s) = \underbrace{\tfrac{1}{11}}_{P_{(c,0)(c,1)}} \cdot s + \underbrace{\tfrac{10}{11}}_{P_{(c,0)(s,0)}}\left(s + \mathbb{E}_{(s,0)} T_1\right)$$

$$= s + \tfrac{10}{11}\mathbb{E}_{(s,0)} T_1$$

So: $\mathbb{E}_{(c,0)} T_1 = \frac{1}{1.1} + \frac{10}{11}\mathbb{E}_{(s,0)} T_1$ $\qquad S_1 \sim \mathcal{E}(q_{(c,0)}) = \mathcal{E}(1.1)$ if $X_0 = (c,0)$

$$\Rightarrow \begin{cases} \mathbb{E}_{(s,0)} T_1 = \boxed{\frac{8}{3}} \longrightarrow \text{answer} \\ \mathbb{E}_{(c,0)} T_1 = \frac{10}{3} \end{cases}$$

In long term, avg rate of fire $\frac{11}{30}$ so expected waiting time until next fire $\approx \frac{30}{11}$, notice that

$$\underbrace{\frac{8}{3}}_{\text{sunny}} < \underbrace{\frac{30}{11}}_{\text{long-term avg}} < \underbrace{\frac{10}{3}}_{\text{cloudy}}.$$